

# STRATEGIES FOR DEDUCTIVE PROOFS

TUTEES 2003/04

## 1. INTRODUCTION

This paper gives you an outline about general strategies concerning deductive proofs in Propositional and Predicate Logic.

The strategies are based on the Tutors<sup>1</sup> own practice in doing proofs for the calculi encountered in Prof. Stephans Lecture Foundations of Logic WS 2003/04. We presuppose knowledge of primitive rules of proof in Propositional and Predicate Logic and wont deal with them in this paper.

You should consider these as a good help in order to do your own proofs.

We could solve at least all homework exercises and mostly all additional exercises provided by the Logic Primer [1] by just using these strategies.

Any logical argument consists of premises and a conclusion. To prove the conclusion you need to apply a finite number of rules and modify your pre-given assumptions until you finally arrive at the conclusion. In most of the proofs you need some extra assumptions on which you work additionally to come to the conclusion (which is nevertheless solely based on the predefined premises).

The strategies may help you to select good extra assumptions.

They do not claim to be complete or adequate strategies for any proof you may encounter in your life.

## 2. PROOF TRICKS

**2.1. Propositional logic.** We use a formal notation to write down possible argument structure. A logical argument consists of a set of premises and one conclusion.

- (1) The syntactical turnstyle  $\vdash$  stands between a set of premises and a conclusion.
- (2) Thus any argment looks like  $\Delta \vdash \phi$   
 $\Delta = \{\delta_1, \dots, \delta_n\}$  is a set of well formed formulas (wffs) in PropLog-  
The premises.  
 $\phi$  is a wff - the conclusion

Lets come to the strategies:

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<sup>1</sup>Boris Bernhardt, Moritz Stephaner, Kerstin Wirz, Michael Gaebler

- (1) Assume each premise (element of  $\Delta$ )  
the first  $n$  lines of your proof will get the following form:

1	(1)	$\delta_1$	$A$
...			
$n$	( $n$ )	$\delta_n$	$A$

- (2) Then look at the structure of the conclusion  $\phi$  of the argument, this will provide you with the information you need for additional assumptions.

**Assumptions you can definitely make without further thinking are denials of the conclusion and all antecedents for conditionals**

But lets look at it more precisely:

- (a) if  $\phi$  is a literal (e.g.  $p$ ) -

Usually a constructive proof (in which we need no RAA) will be possible, so you don't need additional assumptions. If this does not work out, a RAA is still an option.

- (b) if  $\phi$  is a negated literal (e.g.  $\neg p$ ) -

There are no rules to construct negated sentences except for RAA, so you will need at least one in your proof:

Assume  $\phi$  and try to get to a contradiction. If you achieve to get a contradiction resting only on the original assumptions and  $\phi$ , you are almost finished: use RAA to prove  $\neg\phi$  and at the same time get rid of the additional assumption  $\phi$ . This is the general RAA technique, which can in principle be used in any proof - so make sure you get the idea!

- (c) if  $\phi$  is a disjunction ( $\alpha \vee \beta$ ) -

assume  $\neg\phi$  which is  $\neg(\alpha \vee \beta)$ . Try to prove one disjunct of  $\phi$ , now you can easily use  $\vee I$  and RAA. For most proofs of disjuncts, you will need a second RAA (see below)

- (d) if  $\phi$  is a conjunction ( $\alpha \wedge \beta$ )

Try to proof each conjunct according to the given hints and use  $\wedge I$

- (e) if  $\phi$  is a conditional ( $\alpha \rightarrow \beta$ ) -

assume  $\alpha$ .

Try to arrive at  $\beta$  then you can apply  $\rightarrow I$ .

This will discharge  $\alpha$  from your assumption set.

- (f) if  $\phi$  is a biconditional ( $\alpha \leftrightarrow \beta$ ) -

you need (as in math-equivalence proofs) a 2 step proof

(1) assume  $\alpha$ . try to arrive at  $\beta$ . Then you can

Try to arrive at  $\beta$  then you can apply  $\rightarrow I...$

Thus you get  $\alpha \rightarrow \beta$

(2) assume  $\beta$ . try to arrive at  $\alpha$ . Then you can

Try to arrive at  $\alpha$  then you can apply  $\rightarrow I$ ...

Thus you get  $\beta \rightarrow \alpha$

(3) apply  $\leftrightarrow I$  on (1) and (2).

(g) if  $\phi$  is a negated disjunction  $\neg(\alpha \vee \beta)$  - assume  $\neg\alpha$  and  $\neg\beta$  additionally.

(h) if  $\phi$  is a negated conjunction  $\neg(\alpha \wedge \beta)$  - assume  $\alpha$  and  $\beta$  additionally.

In both cases you will probably need 2 *RAAs*

...

You do not need to learn these strategies by heart! It is more important to get an idea of the underlying principles of our proof rules and how you can achieve certain goals in your proof. A good technique is to first write down the premises as assumptions, make some additional assumptions as described above and then hold on and take a close look at the conclusion we want to get. What possible ways are there for the last few steps? Are there important subgoals we will have to achieve? And how can we get these from the given assumptions? Do not start to write until you have rough idea of these intermediate steps. If you desperate with a proof, write down your ideas how you planned the proof. You might still get some points for that.

**2.2. Predicate Logic.** The good thing: Since Propositional Logic is subset of Predicate Logic, we can use all the proof tricks from the preceding section.

But: since Predicate Logic is a more fine grained formal system, this is often not sufficient.

Therefore we need to extend our theory of proof tricks in order to handle the additional types of formulas Predicate Logic Calculus provides.

The crucial property of Predicate Logic proofs is that they are often about quantifier movements and equivalences between different quantified expressions occurring in the formulas.

We cannot access the structure in the scope of a quantifier with our Propositional Logic. In order to capture the Propositional Logic structure of quantified (complex) assumptions we often have to descend as deep as necessary into these formulas to work on them with the Proof Tricks from Propositional Logic.

Descend: We have 2 possibilities for this:

- (1) We cannot access the structure in the scope of a quantifier with our Propositional Logic. We have to descend from quantified formulas to instances:

(a) Universals  $\forall\psi$ 

can be instantiated by  $\forall E$  without any conditions. Of course it is recommended to use the appropriate constants for instantiations.

(b) Existentials  $\exists\psi$ .

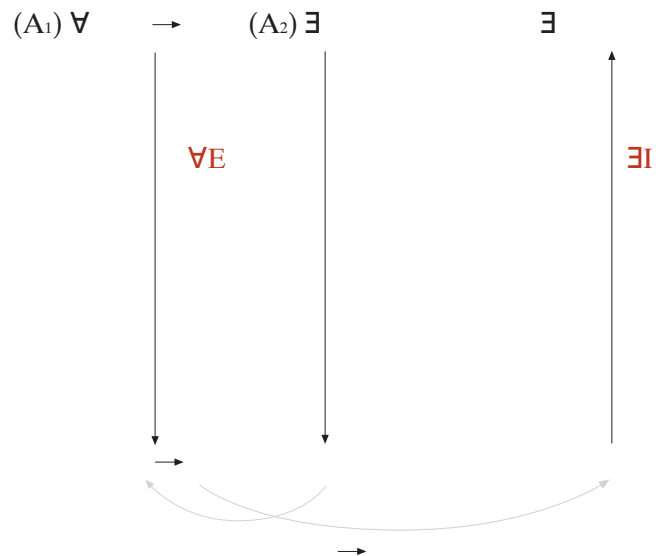
In order to work on these formulas, you have to go an indirect way:

We assume an instance of  $\exists\psi$ , which uses no name already occurring in  $\psi$  and which we did not use already for an instance of a different existential.

Then we can work on the instance. Since we assume something additional we need to get rid of it later. Here comes our friend  $\exists E$  into play:

But step by step.

- (2) Then we let our Proplog system work on disjunctions, conjunctions, negations etc.



- (3) Later on we often have to get onto Quantified Predicate Logical structures again. We have to ascend in the structure from instances to quantified formulas:

(a) Universals  $\forall\psi$ 

A universal quantified formula makes a statement about every object. If we want to quantify over a name in a given formula  $m$ , it has to be the case that this name is just an arbitrary chosen

one. This means that the name over which we want to quantify does not occur in any formulas of ms assumption set,  $\Delta_m$ .

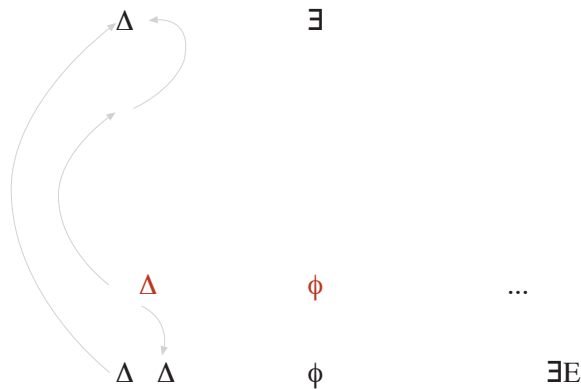
(b) Existentials  $\exists\psi$ .

Here we have got 2 cases:

(i) (1) We like to go from an instance to an Existential Quantified formula in the next line. We can easily do that - no conditions to be respective, since - intuitively - if we know that  $Fa$  we know also that  $\exists Fx$  - This is the famous principle of Extensionality in Logic.

(ii) (2) We like to go from a formula depending on an instance of an Existential to a formula depending on the Existential. Here comes our friend  $\exists E$  into play:

The motivation behind this rule is based on the preceding extra assumed instance of an existential quantified formula. The existential elimination changes the reference (in the assumption set of a formula derived from this extra assumed instance) to the Existential from which its instantiated. Thats it. You just change the reference.



But - and thats crucial - you have to respect the conditions.

The conditions are, that the name of the instance you want to kick out is not allowed to appear in the name you write down again and in its other references than the instance you want to modify. Thus you have to show that the result is not depending on a specific name but on an arbitrary (assumed) one. Thats why you can insert the existential quantified formula into your new assumption set and delete the instance. But step by step.

## 3. REFERENCES

- [1] Colin Allen, Michael Hand. Logic Primer - Second Edition.  
A Bradford Book. The MIT Press. Cambridge. 2002 ( 20/192 Seiten)  
A very straightforward and pragmatic reader with lots of helpful exercises.
- [2] Ansgar Beckermann. Einführung in die Logik.  
Gruyter. 2003 ( 24/382 Seiten)  
A very comprehensive overview about logic within the field of philosophy.  
Unfortunately it does not include syntactic proofs logic and just handles the semantic side.
- [3] Ulrich Nortmann. Sprache, Logik, Mathematik.  
Mentis Verlag. Paderborn. 2003. (24/200 Pages)  
Highly recommended for all of you interested in the "foundations" of logic.
- [4] <http://mitpress.mit.edu/LogicPrimer>  
Link to the LP site with many nice and helpful features.